SECTIONS 3.3 AND 3.4: DERIVATIVE RULES

RECALL: Given a function f, the derivative of f, denoted f', is given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided this limit exists as a real number. If $f'(a)$ exists, then we say f is '*differentiable*' at $x = a$.

NOTATIONS: If $y = f(x)$: $f'(x)$, y', $\dot{y}$, $\dfrac{dy}{dx}$, $\dfrac{d}{dx}[f(x)]$, and $D_x[f(x)]$ all represent the derivative.

In this section, we investigate properties of the derivative. All of these properties can be proved using the definition above. We'll prove a few of them in an 'Appendix' for the interested reader.

PROPERTIES OF THE DERIVATIVE:

Suppose f and g are differentiable and k is a real number constant.

- **The Constant Rule**: $D_x[k] = 0$.
- **The Power Rule**: $D_x[x] = 1$. More generally, $D_x[x^k] = kx^{k-1}$ for $k \neq 0$.
- **The Constant Multiple Rule**: $D_x[kf(x)] = kD_x[f(x)] = kf'(x)$.
- **The Sum/Difference Rule**: $D_x[f(x) \pm g(x)] = D_x[f(x)] \pm D_x[g(x)] = f'(x) \pm g'(x)$.

EXAMPLE 1: Find the indicated derivative.

1. If $f(x) = 3x^2 - 6x + 2$, find $f'(x)$.

$$\begin{aligned}
f'(x) &= D_x[3x^2 - 6x + 2] \\
&= D_x[3x^2] - D_x[6x] + D_x[2] && \text{Sum and Difference Rule} \\
&= 3D_x[x^2] - 6D_x[x] + 0 && \text{Constant Multiple Rule and Constant Rule} \\
&= 3(2x^{2-1}) - 6(1) && \text{Power Rule} \\
f'(x) &= 6x - 6
\end{aligned}$$

2. Find y' if $y = \dfrac{2}{x} - \sqrt{x}$.

   Rewriting y in terms of powers of x, we get $y = 2x^{-1} - x^{\frac{1}{2}}$. Hence,

$$\begin{aligned}
y' &= D_x\left[2x^{-1} - x^{\frac{1}{2}}\right] \\
&= D_x\left[2x^{-1}\right] - D_x\left[x^{\frac{1}{2}}\right] && \text{Sum and Difference Rule} \\
&= 2D_x\left[x^{-1}\right] - \tfrac{1}{2}x^{\frac{1}{2}-1} && \text{Constant Multiple Rule and Power Rule} \\
&= 2(-1)(x^{-1-1}) - \tfrac{1}{2}x^{-\frac{1}{2}} && \text{Power Rule} \\
y' &= -2x^{-2} - \tfrac{1}{2}x^{-\frac{1}{2}}
\end{aligned}$$

3. For $y = \dfrac{6x-1}{2x^3}$, find $\dfrac{dy}{dx}$.

   We first rewrite y in terms of sums and differences of powers:

$$y = \frac{6x-1}{2x^3} = \frac{6x}{2x^3} - \frac{1}{2x^3} = 3x^{-2} - \frac{1}{2}x^{-3}$$

   We now proceed:

$$\begin{aligned}
\frac{dy}{dx} &= D_x\left[3x^{-2} - \frac{1}{2}x^{-3}\right] \\
&= D_x[3x^{-2}] - D_x\left[\frac{1}{2}x^{-3}\right] && \text{Sum and Difference Rule} \\
&= 3D_x[x^{-2}] - \frac{1}{2}D_x\left[x^{-3}\right] && \text{Constant Multiple Rule} \\
&= 3(-2)x^{-2-1} - \frac{1}{2}(-3)x^{-3-1} && \text{Power Rule} \\
\frac{dy}{dx} &= -6x^{-3} + \frac{3}{2}x^{-4}
\end{aligned}$$

EXAMPLE 2: (VIDEO) Find the indicated derivative.

1. For $f(x) = 12x^5 - 3x^2 - 1$, find $f'(x)$.

   Ans: $f'(x) = 60x^4 - 6x$

2. For $y = \dfrac{2x - \sqrt{x}}{5x^2}$, find $\dfrac{dy}{dx}$.

   Ans: $\dfrac{dy}{dx} = -\dfrac{2}{5}x^{-2} + \dfrac{3}{10}x^{-\frac{5}{2}}$

THE PRODUCT AND QUOTIENT RULES

If f and g are differentiable, then:

- **The Product Rule**: $D_x[f(x)g(x)] = D_x[f(x)]g(x) + f(x)D_x[g(x)] = f'(x)g(x) + f(x)g'(x)$.

- **The Quotient Rule**: $D_x\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)D_x[f(x)] - f(x)D_x[g(x)]}{[g(x)]^2} = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$.

EXAMPLE 3: Find the indicated derivative:

1. For $f(x) = (2x-1)(x^2+1)$, find $f'(x)$.

   We have actually two ways to proceed to find $f'(x)$:

   METHOD 1: Simplify the expression for $f(x)$ then take the derivative:

$$\begin{aligned}
f(x) &= (2x-1)(x^2+1) \\
&= 2x^3 - x^2 + 2x - 1 \\
f'(x) &= D_x\left[2x^3 - x^2 + 2x - 1\right] \\
&= D_x\left[2x^3\right] - D_x\left[x^2\right] + D_x\left[2x\right] - D_x\left[1\right] \\
&= 2D_x\left[x^3\right] - 2x^{2-1} + 2D_x[x] - 0 \\
&= 2(3)x^{3-1} - 2x + 2(1) - 0 \\
f'(x) &= 6x^2 - 2x + 2
\end{aligned}$$

   METHOD 2: Use the product rule to take the derivative first, then simplify $f'(x)$.

$$\begin{aligned}
f'(x) &= D_x\left[(2x-1)(x^2+1)\right] \\
&= D_x\left[2x-1\right](x^2+1) + (2x-1)D_x\left[x^2+1\right] && \text{Product Rule} \\
&= \left(D_x[2x] - D_x[1]\right)(x^2+1) + (2x-1)\left(D_x\left[x^2\right] + D_x[1]\right) \\
&= (2-0)(x^2+1) + (2x-1)(2x+0) \\
&= (2)(x^2+1) + (2x-1)(2x) \\
&= 2x^2 + 2 + 4x^2 - 2x \\
f'(x) &= 6x^2 - 2x + 2
\end{aligned}$$

2. If $y = \dfrac{3x-1}{2x+1}$, find $\dfrac{dy}{dx}$.

Here, we are forced to use the Quotient Rule (do you see why?)

$$
\begin{aligned}
\frac{dy}{dx} &= D_x\left[\frac{3x-1}{2x+1}\right] \\
&= \frac{(2x+1)D_x[3x-1]-(3x-1)D_x[2x+1]}{(2x+1)^2} \\
&= \frac{(2x+1)(D_x[3x]-D_x[1])-(3x-1)(D_x[2x]+D_x[1])}{(2x+1)^2} \\
&= \frac{(2x+1)(3-0)-(3x-1)(2+0)}{(2x+1)^2} \\
&= \frac{(2x+1)(3)-(3x-1)(2)}{(2x+1)^2} \\
&= \frac{6x+3-6x+2}{(2x+1)^2} \\
\frac{dy}{dx} &= \frac{5}{(2x+1)^2}
\end{aligned}
$$

EXAMPLE 4: (VIDEO) Find the equation of the tangent line to the graph of $f(x) = \dfrac{x^2}{3x-1}$ at $x = 1$. Check your answer using a graphing utility.

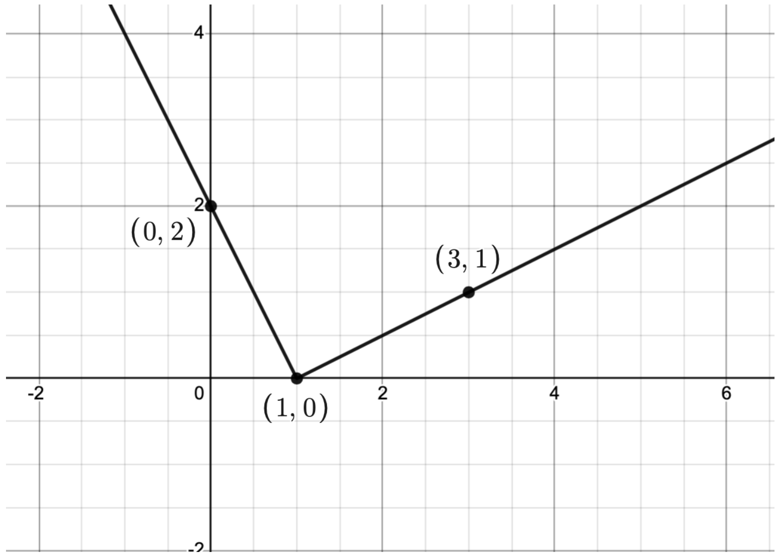
Ans: $y = \dfrac{1}{4}x + \dfrac{1}{4}$

EXAMPLE 5: (VIDEO) Find all places where the graph of $f(x) = 6x - x^3$ has a horizontal tangent line.

Check your answer using a graphing utility.
HINT: Recall that horizontal lines have a slope of 0 . . .

Ans: $x = \pm\sqrt{2}$

EXAMPLE 6: (VIDEO) Suppose $f(x) = 2x - x^2$ and the graph of $y = g(x)$ is given below.



$y = g(x)$

Find the indicated values.

1. $f(0)$, $f'(0)$, $f(3)$ and $f'(3)$.

Ans: $f(0) = 0$, $f'(0) = 2$, $f(3) = -3$ and $f'(3) = -4$.

2. $g(0)$, $g'(0)$, $g(3)$ and $g'(3)$.

Ans: $g(0) = 2$, $g'(0) = -2$, $g(3) = 1$ and $g'(3) = \frac{1}{2}$.

3. $h'(0)$ and $h'(3)$ where $h(x) = 5g(x) - f(x)$.

Ans: $h'(0) = -12$, $h'(3) = \frac{13}{2}$

4. $p'(0)$ and $p'(3)$ if $p(x) = f(x)\,g(x)$.

Ans: $p'(0) = 4$, $p'(3) = -\frac{11}{2}$

5. $q'(0)$ and $q'(3)$ if $q(x) = \dfrac{f(x) - 2}{g(x)}$.

Ans: $q'(0) = 0$, $q'(3) = -\frac{3}{2}$

HIGHER ORDER DERIVATIVES: There is no reason we cannot take derivatives of derivatives. Below we list some different notations for higher order derivatives.

- The **second** derivative is the derivative of the (first) derivative:

  If $y = f(x)$, notations for the second derivative include: $f''(x)$, y'', $\dfrac{d^2y}{dx^2}$, $\dfrac{d^2}{dx^2}[f(x)]$, and $D_x^2[f(x)]$

- The **third** derivative is the derivative of the second derivative:

  If $y = f(x)$, notations for the third derivative include: $f'''(x)$, y''', $\dfrac{d^3y}{dx^3}$, $\dfrac{d^3}{dx^3}[f(x)]$, and $D_x^3[f(x)]$

- The **fourth** derivative is the derivative of the third derivative:

  If $y = f(x)$, notations for the third derivative include: $f^{(4)}(x)$, $y^{(4)}$, $\dfrac{d^4y}{dx^4}$, $\dfrac{d^4}{dx^4}[f(x)]$, and $D_x^4[f(x)]$

- In general, the nth derivative is denotes as: $f^{(n)}(x)$, $y^{(n)}$, $\dfrac{d^ny}{dx^n}$, $\dfrac{d^n}{dx^n}[f(x)]$, and $D_x^n[f(x)]$

DEFINITION: If $s(t)$ represents a position function, then we know $s'(t) = v(t)$ represents the velocity function. The function $s''(t) = v'(t) = a(t)$ is the **acceleration** function. In accordance to Newton's Laws of Motion, the force acting on an object is directly proportional the acceleration of the object.

EXAMPLE 7: The function $s(t) = -5t^2 + 100t$ for $0 \le t \le 20$ gives the height, in feet, of a model rocket above the Moon's surface t seconds after lift-off. Find and interpret the acceleration function $a(t)$.
We start finding $a(t) = s''(t)$ by finding $s'(t)$.

$$s'(t) = D_t\left[-5t^2 + 100t\right] = D_t\left[-5t^2\right] + D_t\left[100t\right] = -5D_t\left[t^2\right] + 100D_t\left[t\right] = -5(2t) + 100 = -10t + 100.$$

Hence,

$$a(t) = s''(t) = D_t[s'(t)] = D_t[-10t + 100] = D_t[-10t] + D_t[100] = -10D_t[t] - 0 = -10.$$

To interpret what this means, we first need to determine the accurate units.
Since $s(t)$ is measured in feet and t is measured in seconds, $s'(t) = \dfrac{ds}{dt}$ has units $\dfrac{\text{feet}}{\text{second}}$.
Hence, $s''(t) = \dfrac{d}{dt}[s'(t)]$ has units $\dfrac{\text{feet/second}}{\text{second}} = \dfrac{\text{feet}}{\text{second}^2}$.
Since $a(t) = -10$, the rocket experiences a constant acceleration of 10 feet per second squared **downwards**.

NOTE: This means the rocket's **velocity** is always **decreasing**. Hence, when the rocket is traveling **upwards**, the rocket is **slowing down** since the acceleration is **opposite** the direction of the motion; when the rocket is traveling **downwards**, the rocket is **speeding up** since the acceleration is in the **same direction** as the motion.

HOMEWORK: Section 3.3: 19 - 79 every other odd; Section 3.4: 19 - 83 every other odd

APPENDIX: SELECTED PROOFS OF DERIVATIVE PROPERTIES

The proofs use the definition of derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

the limit laws, and the notion of local linearity:

If f is differentiable at $x = a$, then near $x = a$, $f(x) \approx f'(a)(x-a) + f(a)$.

- **The Constant Rule**: $D_x[k] = 0$.

  PROOF: Let $f(x) = k$. Then $f(x+h) = k$, so

  $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$$

- **The Power Rule**: $D_x[x] = 1$. More generally, $D_x[x^k] = kx^{k-1}$ for $k \neq 0$.

  PROOF: If $f(x) = x$, then $f(x+h) = x+h$ so:

  $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

  If $k \geq 2$ is a natural number, then the Binomial Theorem says: $(x+h)^k = x^k + k\,x^{k-1}\,h$+terms with a factor of at least h^2.

  Hence, $(x+h)^k = x^k + k\,x^{k-1}\,h + h^2\,(\text{stuff})$. In this case, we get:

  $$\begin{aligned}
  f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
  &= \lim_{h\to 0} \frac{(x+h)^k - x^k}{h} \\
  &= \lim_{h\to 0} \frac{x^k + k\,x^{k-1}\,h + h^2\,(\text{stuff}) - x^k}{h} \\
  &= \lim_{h\to 0} \frac{k\,x^{k-1}\,h + h^2\,(\text{stuff})}{h} \\
  &= \lim_{h\to 0} [k\,x^{k-1} + h\,(\text{stuff})] \\
  &= k\,x^{k-1}
  \end{aligned}$$

  For negative powers $x^{-k} = \dfrac{1}{x^k}$ we can verify the power rule using the quotient rule.

  For more general powers such as fractional exponents, stay tuned! We'll return to this in a later section.

- **The Constant Multiple Rule**: $D_x[kf(x)] = kD_x[f(x)] = kf'(x)$.

  PROOF:

  $$D_x[kf(x)] = \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h} = \lim_{h\to 0} \frac{k[f(x+h) - f(x)]}{h} = k \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = kf'(x)$$

- **The Sum/Difference Rule**: $D_x[f(x) \pm g(x)] = D_x[f(x)] \pm D_x[g(x)] = f'(x) \pm g'(x)$.

  PROOF:

$$\begin{aligned} D_x[f(x) \pm g(x)] &= \lim_{h\to 0} \frac{[f(x+h) \pm g(x+h)] - [f(x) \pm g(x)]}{h} \\ &= \lim_{h\to 0} \frac{[f(x+h) - f(x)] \pm [g(x+h) - g(x)]}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) \pm g'(x) \end{aligned}$$

- **The Product Rule**: $D_x[f(x)g(x)] = D_x[f(x)]g(x) + f(x)D_x[g(x)] = f'(x)g(x) + f(x)g'(x)$.

  PROOF: If f and g are differentiable at $x = a$, local linearity says that near $x = a$

$$f(x) \approx f'(a)(x-a) + f(a) \quad \text{and} \quad g(x) \approx g'(a)(x-a) + g(a).$$

Relabeling $x = a + h$ (so $x - a = h$), we get $f(a+h) \approx f'(a)h + f(a)$ and $g(a+h) \approx g'(a)h + g(a)$. Hence near $x = a$,

$$f(a+h)g(a+h) \approx [f'(a)h + f(a)][g'(a)h + g(a)] = f'(a)g'(a)h^2 + f'(a)g(a)h + f(a)g'(a)h + f(a)g(a)$$

Putting it all together, we get:

$$\begin{aligned} (fg)'(a) &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\ &= \lim_{h\to 0} \frac{f'(a)g'(a)h^2 + f'(a)g(a)h + f(a)g'(a)h + f(a)g(a) - f(a)g(a)}{h} \\ &= \lim_{h\to 0} \frac{f'(a)g'(a)h^2 + f'(a)g(a)h + f(a)g'(a)h}{h} \\ &= \lim_{h\to 0} \frac{h[f'(a)g'(a)h + f'(a)g(a) + f(a)g'(a)]}{h} \\ &= \lim_{h\to 0} [f'(a)g'(a)h + f'(a)g(a) + f(a)g'(a)] \\ &= f'(a)g(a) + f(a)g'(a) \end{aligned}$$

NOTE: The Quotient Rule can be proved similarly (though the algebra is a bit messier.)